Chapter 4

Determination of Local and Global Weights of Alternatives from Inconsistent Interval Judgment Matrices

4.1 Introduction

Saaty (1977a) proposes the entries of pairwise comparison matrices as single numbers, e.g., if one alternative is strongly preferred relative to another, then the ratio assigned is 5. But in the cases where fuzziness or randomness or any other kind of uncertainty is concerned, DM may prefer to provide his judgments by means of range of numbers based on the same (1/9-9) scale (Saaty and Vargas, 1987; Arbel, 1989; Salo, 1993). For example, in the foregoing case, the range may be [4,6] with midpoint 5 as the most probable value. But the problem is how to extract weights from such a set of responses.

Arbel (1989) formulated the problem of determination of weights from interval judgments as a linear programming problem (LPP). But the LP approach gives solution only when the matrices are consistent. Salo and Hämäläinen (1995) extended Arbel's LP approach from one level of hierarchy to the entire hierarchy to obtain weight interval for each of the decision alternatives. By using LP approach, Salo (1993) determined weights of alternatives from inconsistent interval matrices by increasing the length of intervals. Saaty and Vargas (1987) introduced the notion of simulation in AHP to investigate the effect of uncertainty in judgments on the stability of rank order of alternatives. In this chapter, we determine the local as well as global weights of the alternatives by simulation technique based upon three different probability distributions, namely, uniform, truncated normal, and truncated gamma, where the comparison matrices are inconsistent and the elements of the matrices are presented in the form of closed intervals. Subsequently, the effect of these distributions on the two sets of weights is investigated by means of a statistical analysis.

4.2 Interval Judgments

The reasons for adopting interval judgments to construct pairwise comparison matrices in AHP may be summarized as:

- probabilistic uncertainty in the decision making environment (Saaty and Vargas, 1987),

- fuzzy uncertainty in the decision making environment (fuzzy numbers can also be employed in this case) (Van Laarhoven and Pedrycz, 1983; Buckley, 1985),

- incomplete information (Arbel, 1989; Arbel and Vargas, 1993),

- group decision (Hämäläinen et al., 1992),
- unfamiliarity with the decision making process,
- to minimize preference elicitation time,
- to avoid risk in giving point estimate in political decision making, and
- unwillingness to specify point judgments due to any other reason.

In the foregoing cases, a comparison matrix takes the form:

$$\mathbf{A'} = \begin{array}{|c|ccccc|} \hline & O_1 & O_2 & O_3 & \dots & O_n \\ \hline O_1 & 1 & [l_{12}, u_{12}] & [l_{13}, u_{13}] & \dots & [l_{1n}, u_{1n}] \\ O_2 & & 1 & [l_{23}, u_{23}] & \dots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ O_n & & & & & 1 \\ \hline \end{array} \tag{4.1}$$

Where O_i, i = 1, 2,…, n, are the objects which are compared and $l_{ij} \leq w_i / w_j \leq u_{ij}$, i, j = 1, 2,…, n, $\mathbf{w} = (w_1, w_2, ..., w_n)^T$ being the weight vector. The respondent should fill up only the upper triangular part of (4.1). The lower triangular part is to be constructed by

$$\begin{aligned} l_{ij} &= 1/u_{ji}, \qquad i = 2,3,...,n; \quad j = 1,2,....,i-1 \\ u_{ij} &= 1/l_{ji}, \qquad i = 2,3,...,n; \quad j = 1,2,...,i-1 \end{aligned}$$

Note that due to the procedure of construction of (4.1), the relations $l_{ij}\, l_{ji} < 1$ and $u_{ij}\, u_{ji} > 1$, hold for all i and j.

Definition 4.1: The length of an interval $[l_{ij}, u_{ij}]$ of the pairwise comparison matrix $\boldsymbol{A'}$ in (4.1) may be defined as:

$$\begin{aligned} &u_{ij} - l_{ij}, \quad \text{when} \quad l_{ij} \geq 1, \quad u_{ij} \geq 1 \\ &l_{ij}^{-1} - u_{ij}^{-1}, \quad \text{when} \quad l_{ij} < 1, \quad u_{ij} < 1 \\ &u_{ij} + l_{ij}^{-1} - 2, \quad \text{when} \quad l_{ij} < 1, \quad u_{ij} \geq 1 \end{aligned}$$

4.3 Determination of Weights from Interval Judgments by Simulation

Usually, in the simulation approach, the weights are elicited from $\boldsymbol{A'}$ in (4.1) on replacing the intervals by means of random numbers, belonging to respective intervals, which follow a specific probability distribution. The transformed matrix is, in general, not a consistent one. If it is a consistent matrix and $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ represent independent random variables for the components of weight vector, then the joint probability density function is given by

$$f(X_1, X_2, \ldots, X_n) = \frac{\prod_{i=1}^{n} \beta_i^{\alpha_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \prod_{i=1}^{n} X_i^{\alpha_i - 1} e^{-\beta_i x_i}, \quad X_i, \alpha_i, \beta_i > 0 \tag{4.2}$$

assuming the matrix entries are gamma distributed with parameters α_i and β_i. By means of a suitable transformation, Vargas (1982) has reduced (4.2) to

$$g_n(Y_1, Y_2, \ldots, Y_n) \prod_{i=1}^{n} [\frac{1}{\Gamma(\alpha_i)} Y_i^{\alpha_i - 1}] \, (1 - \sum_{k=1}^{n-1} Y_k)^{\alpha_n - 1}. \; Y_n^{\sum_{k=1}^{n} \alpha_k - 1}. \quad e^{-Y_n} \tag{4.3}$$

Hence, the marginal distribution of ($Y_1, Y_2, \ldots, Y_{n-1}$) is

$$g_{n-1}(Y_1, Y_2, \ldots, Y_{n-1}) = \frac{\gamma(\alpha_1 + \alpha_2 + \ldots + \alpha_n)}{\prod_{i=1}^{n} \gamma(\alpha_i)} \prod_{k=1}^{n-1} Y_k^{a_k - 1} (1 - \sum_{k=1}^{n-1} Y_k) \tag{4.4}$$

where $0 < Y_k < 1$, $k = 1, 2, \ldots, n-1$, and $\sum_{k=1}^{n-1} Y_k < 1$. This shows that the marginal distribution of ($Y_1, Y_2, \ldots, Y_{n-1}$) follows a Dirichlet's distribution with parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$. It can be shown that the individual variable Y_k, $k = 1, 2, \ldots, n-1$, follows beta distribution with parameters ($\alpha_k, \sum_{i=1}^{n} \alpha_i$).

Khane (1975), in his pioneering work, stated that although the distribution of the random variable belonging to some interval is to be determined from the available data, but with a lack of sufficient information, one may assume that the interval [l_{ij}, u_{ij}] represents a grade somewhere between l_{ij} and u_{ij}, any number in that region being equally probable. In the same paper, he also mentioned that for some problems, normal or any other distribution might also be appropriate. But the question arises what will be the probability distribution of components of principal eigenvector when the matrix entries will follow truncated normal or any other truncated distribution. Analytical derivation of the distribution is a bit difficult due to the complexity of eigenvector calculation procedure. To investigate the effect of various probability distributions on the overall weights experimentally, three probability distributions, viz., uniform, truncated normal and truncated gamma, are considered in the simulation approach. For this purpose, we have considered a multi-criteria financial investment problem where the length of interval in each cell is unity. In order to observe the variation of the local as well as global weights of the alternatives (here portfolios), the computation has been repeated by successively increasing the common length of all the intervals from 1 through 5 units. Although, for complete uncertainty, the range is [1/9-9] (i.e., the length of interval is 16), according to the interpretation of the ratings of the (1/9-9) scale, the interval length 5 (for instance, [1,6], where 1 and 6, respectively, represent equal preference and more than strong preference) is large enough to capture DM's wavering mind. Moreover, the weights for interval lengths more than 5 can be extrapolated from Table 4.2. In addition to this, Saaty and Vargas (1987, page 109) writes: *"...large ambiguity in the judgment can render ranking a useless pursuit"*.

In the uncertain environment, although DM may not be sure about a point judgment, his mind may, however, waver about that point. This very concept has been used in fuzzy AHP (see Van Laarhoven and Pedrycz, 1983). Following this trend of mind of the DM, in our experiment, we have considered middle points of the intervals as means of normal and gamma distributions. Further, the variance depends upon the decision maker's perception about a particular problem. It is well known that variance of uniform distribution is $(u_{ij} - l_{ij})^2/12$. In the present experiment, to compare the results with uniform distribution, same variance has been considered for normal and gamma distributions.

The algorithm in the computation is as follows:

Step 1. Construct the hierarchy of the problem.
Step 2. Construct the comparison matrices involving interval judgments.
Step 3. Adopting a suitable random number generator, replace the intervals in step 2 by single numbers.
Step 4. Using the eigenvector method, elicit local priority weights of the alternatives.
Step 5. Repeat steps 3 and 4. After N (N>1000, say) such repetitions take simple average of the weights.
Step 6. To estimate the global weights, use principle of hierarchical composition (Saaty, 1977a).

4.4 A Multiple Criteria Decision Making problem

Suppose, a person is interested to invest his money in any one of the four portfolios: Bank Deposit (BD), Debentures (DB), Government Bond (GB), and Shares (SH). Out of these four portfolios, he has to choose only one based upon the criteria: Return (Re), Risk (Ri), Tax Benefits (Tb), and Liquidity (Li). This is a problem where uncertainty is inherently associated (Saaty, 1987a). The hierarchy of the problem is shown in Fig. 4.1.

The pairwise comparison matrices for all criteria as well as for all the alternatives are constructed in consultation with some experts whose major research area is Financial Management. The elements of the comparison matrices are shown in Table 4.1.

As mentioned in the foregoing algorithm, each interval is to be replaced by a single number, which lies within that interval. Here these numbers (which are random) have been generated by using the subroutines RNU, RNNOR, RNGAM from IMSL in CYBER 180/840A. The numbers in the lower triangular part of each matrix are the reciprocal of the corresponding numbers in the upper triangular part. The cells in the pairwise comparison matrices are not independent from each other, i.e., at the time of filling up the matrices, content of one cells depends upon the contents of other cells. In the present case, this dependence has already been taken into account at the time of filling up the matrices by means of range of numbers.

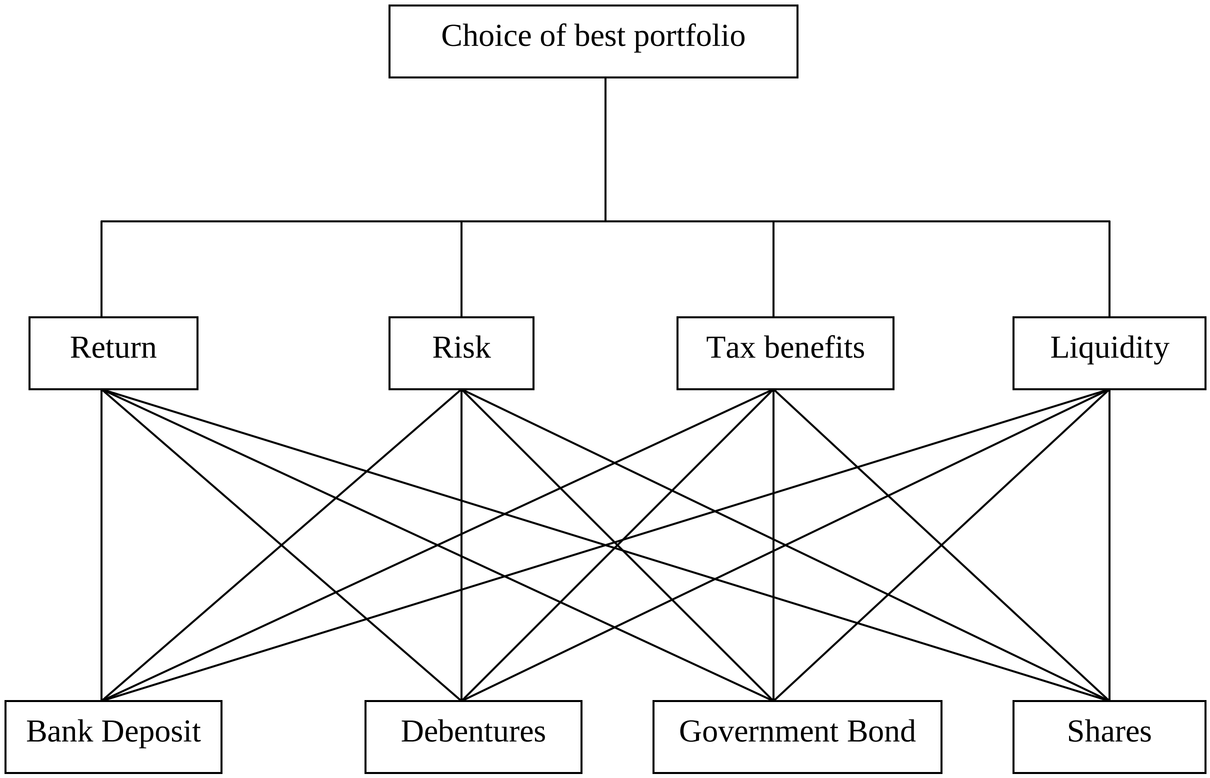

FIG. 4.1: Hierarchy of the portfolio selection problem.

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Table 4.1: Pairwise comparison matrices of the portfolio selection problem

	Re	Ri	Tb	Li
Re	1	[3,4]	[5,6]	[6,7]
Ri		1	[4,5]	[5,6]
Tb			1	[3,4]
Li				1

Re	BD	DB	GB	SH	Ri	BD	DB	GB	SH
BD	1	[1/4,1/3]	[3/4]	[1/6,1/5]	BD	1	[3,4]	[4,5]	[6,7]
DB		1	[6,7]	[1/5,1/4]	DB		1	[3,4]	[5,6]
GB			1	[1/7,1/6]	GB			1	[4,5]
SH				1	SH				1

Tb	BD	DB	GB	SH	Li	BD	DB	GB	SH
BD	1	1	[1/6,1/5]	[1/4,1/3]	BD	1	[3,4]	6	[6,7]
DB		1	[1/6,1/5]	[1/4,1/3]	DB		1	[3,4]	[3,4]
GB			1	[4,5]	GB			1	[3,4]
SH				1	SH				1

We follow the foregoing algorithm assuming at first uniform distribution for the random variable in each of the intervals of the five matrices in table 4.1. In order to observe the behavior of the components of eigenvector as well as the variation of the global weights of the alternatives, we perform the experiment by varying simulation run size from N = 1000 through 5000. For the same purpose, the intervals have been changed to $[l_{ij}-0.5, u_{ij}+0.5]$, $[l_{ij}-1.0, u_{ij}+1.0]$, and $[l_{ij}-2.0, u_{ij}+2.0]$, $l_{ij}, u_{ij} > 1$, for all $i = 1, 2, ..., n-1$; $j = i+1, i+2, ..., n$ (for $l_{ij}, u_{ij} < 1$, changes are made in the denominators), where the respective lengths of the changed intervals are 2, 3, and 5 units, respectively. We then repeat the experiment for these three cases. Further, the entire experiment has been repeated by considering the probability distributions, namely, truncated normal and truncated gamma for the random variables. The weights of the criteria and overall weights of the four alternatives for all the cases are shown in Table 4.2 and Table 4.3, respectively.

Table 4.2: Relative weights of the criteria for various probability distributions and various simulation run sizes

		Return				Risk			
Simulation Size Run	Prob. Dist.	Interval Length				Interval Length			
		1	2	3	5	1	2	3	5
1000	Uniform	0.5674	0.5663	0.5630	0.5561	0.2775	0.2773	0.2801	0.2830
	T. Normal	0.5686	0.5674	0.5660	0.5552	0.2764	0.2771	0.2780	0.2858
	T. Gamma	0.5681	0.5669	0.5632	0.5545	0.2768	0.2772	0.2794	0.2821
2000	Uniform	0.5675	0.5651	0.5640	0.5533	0.2772	0.2785	0.2789	0.2855
	T. Normal	0.5683	0.5666	0.5670	0.5589	0.2767	0.2777	0.2769	0.2819
	T. Gamma	0.5680	0.5657	0.5631	0.5522	0.2767	0.2783	0.2792	0.2852
3000	Uniform	0.5677	0.5664	0.5641	0.5534	0.2771	0.2776	0.2788	0.2849
	T. Normal	0.5683	0.5674	0.5656	0.5581	0.2767	0.2771	0.2789	0.2824
	T. Gamma	0.5680	0.5660	0.5631	0.5530	0.2768	0.2779	0.2794	0.2848
4000	Uniform	0.5681	0.5666	0.5638	0.5545	0.2766	0.2776	0.2791	0.2845
	T. Normal	0.5683	0.5666	0.5658	0.5590	0.2767	0.2777	0.2780	0.2821
	T. Gamma	0.5678	0.5659	0.5631	0.5539	0.2770	0.2780	0.2793	0.2835
5000	Uniform	0.5677	0.5663	0.5642	0.5553	0.2770	0.2777	0.2790	0.2829
	T. Normal	0.5680	0.5673	0.5646	0.5591	0.2768	0.2771	0.2791	0.2819
	T. Gamma	0.5678	0.5662	0.5628	0.5516	0.2770	0.2778	0.2797	0.2855

Table 4.2 continued

Simulation Size Run	Prob. Dist.	Tax Benefits: Interval Length 1	2	3	5	Liquidity: Interval Length 1	2	3	5
1000	Uniform	0.1041	0.1050	0.1052	0.1069	0.0510	0.0513	0.0517	0.0540
	T. Normal	0.1040	0.1041	0.1045	0.1059	0.0511	0.0513	0.0515	0.0531
	T. Gamma	0.1041	0.1045	0.1051	0.1081	0.0511	0.0514	0.0523	0.0552
2000	Uniform	0.1043	0.1050	0.1052	0.1068	0.0510	0.0514	0.0520	0.0544
	T. Normal	0.1041	0.1044	0.1046	0.1061	0.0509	0.0512	0.0516	0.0531
	T. Gamma	0.1044	0.1045	0.1055	0.1077	0.0510	0.0515	0.0523	0.0549
3000	Uniform	0.1042	0.1046	0.1052	0.1073	0.0510	0.0514	0.0519	0.0545
	T. Normal	0.1041	0.1044	0.1047	0.1067	0.0509	0.0512	0.0516	0.0528
	T. Gamma	0.1041	0.1047	0.1052	0.1076	0.0510	0.0515	0.0522	0.0548
4000	Uniform	0.1042	0.1045	0.1051	0.1069	0.0510	0.0513	0.0519	0.0540
	T. Normal	0.1041	0.1046	0.1048	0.1060	0.0509	0.0511	0.0514	0.0529
	T. Gamma	0.1042	0.1047	0.1053	0.1076	0.0510	0.0514	0.0523	0.0550
5000	Uniform	0.1043	0.1047	0.1048	0.1073	0.0510	0.0513	0.0519	0.0545
	T. Normal	0.1042	0.1040	0.1047	0.1061	0.0509	0.0512	0.0516	0.0530
	T. Gamma	0.1041	0.1045	0.1053	0.1078	0.0510	0.0515	0.0522	0.0550

Table 4.3: Relative weights of the alternatives for various probability distributions and various simulation run sizes

Simulation Size Run	Prob. Dist.	Bank Deposits: Interval Length 1	2	3	5	Debentures: Interval Length 1	2	3	5
1000	Uniform	0.2521	0.2518	0.2537	0.2543	0.2346	0.2350	0.2353	0.2343
	T. Normal	0.2520	0.2519	0.2526	0.2553	0.2345	0.2350	0.2355	0.2370
	T. Gamma	0.2520	0.2522	0.2527	0.2536	0.2340	0.2341	0.2350	0.2365
2000	Uniform	0.2521	0.2526	0.2531	0.2563	0.2343	0.2350	0.2353	0.2371
	T. Normal	0.2519	0.2527	0.2517	0.2545	0.2342	0.2346	0.2350	0.2369
	T. Gamma	0.2518	0.2526	0.2529	0.2555	0.2340	0.2347	0.2348	0.2376
3000	Uniform	0.2522	0.2521	0.2527	0.2556	0.2340	0.2347	0.2353	0.2381
	T. Normal	0.2519	0.2519	0.2525	0.2542	0.2341	0.2343	0.2349	0.2363
	T. Gamma	0.2519	0.2525	0.2531	0.2550	0.2341	0.2344	0.2352	0.2375
4000	Uniform	0.2518	0.2523	0.2528	0.2552	0.2343	0.2346	0.2354	0.2375
	T. Normal	0.2519	0.2523	0.2526	0.2546	0.2341	0.2344	0.2345	0.2359
	T. Gamma	0.2520	0.2524	0.2532	0.2554	0.2341	0.2342	0.2354	0.2371
5000	Uniform	0.2520	0.2523	0.2531	0.2542	0.2341	0.2346	0.2352	0.2379
	T. Normal	0.2519	0.2520	0.2528	0.2540	0.2341	0.2344	0.2349	0.2370
	T. Gamma	0.2519	0.2525	0.2534	0.2557	0.2342	0.2344	0.2353	0.2378

Table 4.3 continued

Simulation Size Run	Prob. Dist.	Government Bond Interval Length 1	2	3	5	Shares Interval Length 1	2	3	5
1000	Uniform	0.1323	0.1331	0.1337	0.1371	0.3810	0.3800	0.3774	0.3715
	T. Normal	0.1321	0.1327	0.1328	0.1361	0.3814	0.3804	0.3791	0.3716
	T. Gamma	0.1324	0.1331	0.1345	0.1380	0.3817	0.3805	0.3778	0.3718
2000	Uniform	0.1325	0.1334	0.1340	0.1374	0.3811	0.3798	0.3778	0.3691
	T. Normal	0.1323	0.1327	0.1330	0.1354	0.3815	0.3800	0.3803	0.3733
	T. Gamma	0.1326	0.1331	0.1344	0.1382	0.3817	0.3794	0.3780	0.3688
3000	Uniform	0.1323	0.1330	0.1340	0.1372	0.3814	0.3771	0.3780	0.3691
	T. Normal	0.1323	0.1327	0.1333	0.1358	0.3816	0.3811	0.3793	0.3737
	T. Gamma	0.1325	0.1330	0.1343	0.1382	0.3815	0.3801	0.3772	0.3694
4000	Uniform	0.1324	0.1329	0.1339	0.1374	0.3814	0.3803	0.3779	0.3695
	T. Normal	0.1323	0.1328	0.1332	0.1353	0.3817	0.3805	0.3797	0.3742
	T. Gamma	0.1325	0.1332	0.1343	0.1383	0.3814	0.3802	0.3771	0.3692
5000	Uniform	0.1325	0.1331	0.1337	0.1373	0.3813	0.3800	0.3779	0.3707
	T. Normal	0.1323	0.1327	0.1334	0.1354	0.3815	0.3808	0.3790	0.3737
	T. Gamma	0.1324	0.1331	0.1342	0.1384	0.3814	0.3800	0.3771	0.3680

4.5 A Statistical Analysis of the Results

In this section, we statistically investigate the effects of various probability distributions, various interval lengths, and various simulation run sizes on the local (Table 4.2) and global (Table 4.3) set of weights. It is worth mentioning that the problem of investigation by pure mathematical or analytical approach is intractable (Vargas, 1982; Saaty and Vargas, 1987; Jimenez and Vargas, 1993). Empirical investigation is the only way left before us.

It is to be noted that by virtue of the Central Limit Theorem, the weights computed by the algorithm of the previous section follow normal distribution. To test the significance of difference among various levels of the three factors, namely, distribution, interval length and simulation run size, we perform a 3-way Analysis of Variance taking all the interaction terms into account. The analysis has been carried out for the data in both Table 4.2 and Table 4.3. But we provide the F-ratios in Table 4.4 for various cases with respect to the data in Table 4.2 only.

We note that for all the criteria there is a significant difference among various distributions and interval lengths, whereas simulation run sizes are not significantly different. Exactly similar analysis follows for data in Table 4.3; but in this case, it is observed that the magnitudes of the F-ratios have been decreased.

It was stated previously that truncated normal random numbers were generated considering middle points of intervals as means with some common variance. For normal distribution, the average of large number of sample points tends to its mean. So, by virtue of the property of stability of eigenvector components, the two sets of weights obtained by using normal distribution and replacing the intervals by means of their respective mid points should be close to each other. This is obvious from the weights of the four portfolios: Bank Deposit, Debentures, Government Bond, and Shares, viz., 0.2519, 0.2340, 0.1323, and 0.3819, respectively, obtained by replacing the intervals by means of their respective mid-points and the corresponding weights 0.2519, 0.2343, 0.1327, and 0.3811 obtained by using normal distribution as shown in Table 4.3 (for interval length 2 and simulation run size 3000).

Table 4.4: F-ratios for the three factors with respect to the criteria matrix

Criteria	Factor	Computed F-ratio	Tabulated F-ratio (5%)
Return	Distribution	36.11	3.40
	Interval length	626.83	3.01
	Simulation run size	0.23	2.78
Risk	Distribution	5.96	3.40
	Interval length	177.84	3.01
	Simulation run size	0.09	2.78
Tax benefits	Distribution	58.64	3.40
	Interval length	658.02	3.01
	Simulation run size	0.61	2.78
Liquidity	Distribution	270.33	3.40
	Interval length	2637.36	3.01
	Simulation run size	1.34	2.78

Remark 4.1: While generating random numbers in the experiment, for simplicity, same probability distribution has been assumed for all the cells of a pairwise comparison matrix. Of course, in reality, this may not necessarily be true. In some interval it may be uniform, in another it may be any other distribution.

4.6 Concluding Remarks

Articulation of interval judgments is a flexible way to express one's preference strength in an uncertain environment. A simulation experiment has been performed to investigate the effect of various probability distributions for the random variables belonging to the interval judgments on the local as well as global sets of weights of the alternatives. Gradual increment in the interval length and simulation run size has been made to study the behavior of the varied weights. A detailed statistical analysis of the two sets of weights shows significant difference among various distributions and various interval lengths and insignificant difference in various simulation run sizes. Lastly, we emphasize that the conclusions, which have been drawn from the experiment involving only four criteria and four alternatives, are independent of the size of comparison matrices.